Data Types with Symmetries via Action Containers

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Overview

Goal of the talk

Introduce *action containers* to model data types with symmetries

Contents

- Background
  - Endofunctors and algebraic data types
  - Containers for polynomial functors
- Action containers
  - Construction via universal property
  - Closure properties
- 2-categorical interpretation: Embedding as 2-endofunctors of groupoids

Endofunctors model algebraic data types

Many data types can be modeled in the category of **Set**-endofunctors:

$$\mathrm{List}(X) := \sum_{n:\mathbb{N}} (\mathrm{Fin}(n) \to X) \quad \mathrm{Maybe}(X) := 1 + X \quad \mathrm{RoseTree}(X) := \mu Y.\, X + \mathrm{List}(Y)$$

New endofunctors can be built by "algebraic" manipulations:

"a pair of an F and a G"	"either an F or a G"	"a K-tuple of Fs"
$F \times G$	$F + G$	F^K

Containers: syntax for polynomials

The nice class of *polynomial* endofunctors is described by *containers*:

a container

$$(S \triangleleft P)$$

$$S : \mathbf{Set}, P : S \to \mathbf{Set}$$

its interpretation as a polynomial

$$[\![S \triangleleft P]\!](X) := \sum_{s:S} (P(s) \to X)$$

Sanity Check

$[\![-]\!] : \mathbf{Container} \to \mathrm{Endo}(\mathbf{Set})$ is a fully faithful functor.

Morphisms of containers describe *exactly* the morphisms of their interpretations.

Non-polynomial endofunctors

Caveat

Not all interesting functors are covered by this framework.

Example

Finite multisets are not polynomial:

$$\mathsf{FMSet}(X) := \sum_{n:\mathbb{N}} (\mathsf{Fin}(n) \to X) / \sim$$

The relation is generated by identifying tuples up to permutation:

$$(x_1, \ldots, x_n) \sim (x_{\pi 1}, \ldots, x_{\pi n}) \quad \forall \pi : \mathsf{Fin}(n) \simeq \mathsf{Fin}(n)$$

Action containers

Definition

An action container $F = (S \triangleright P \triangleleft^{\sigma} G)$ consists of

shapes	a set S
positions	a family of sets $P : S \to \mathbf{Set}$
symmetries	a family of groups $G : S \to \mathbf{Group}$
actions	a family of group actions: for each $s : S$, σ_s is an action of G_s on P_s

Intuition

Symmetries tell us under which permutations of positions the data type is invariant.

Interpretation

$$[\![S \triangleright P \triangleleft^{\sigma} G]\!](X) := \sum_{s:S} (P_s \to X)/\sim_s \qquad v \sim_s w := \exists g : G_s.\, v = w \circ \sigma_s(g)$$

Group actions?

Definition

An action of a group G on a set X is a group homomorphism $\sigma : G \to \mathfrak{S}(X)$, where $\mathfrak{S}(X)$ is the group of *automorphisms* $X \simeq X$.

Equivalently, a G-action on X is a functor $\mathbb{B}\sigma : \mathbb{B}G \to \mathrm{Set}$

- $\mathbb{B}G$ is G seen as a 1-object groupoid
- the lone object $\bullet$ is sent to X
- loops $g : \bullet \xrightarrow{\sim} \bullet$ are sent to $\sigma(g) : X \xrightarrow{\sim} X$

Example: Finite multisets

Finite multisets are "lists up to permutation", thus come from *the* permutation action:

$$\mathsf{FMSet} = (n : \mathbb{N} \triangleright \mathsf{Fin}(n) \triangleleft^{\mathsf{id}} \mathfrak{S}(\mathsf{Fin}(n))) \qquad \mathsf{id} : \mathfrak{S}(\mathsf{Fin}(n)) \to \mathfrak{S}(\mathsf{Fin}(n))$$

Its interpretation:

$$[\![\mathsf{FMSet}]\!]X = \sum_{n:\mathbb{N}} X^n/{\sim}$$

where $(\sim)$ is generated by

$$(x_1, \ldots, x_n) \sim (x_{\pi(1)}, \ldots, x_{\pi(n)}) \quad \forall \pi : \mathsf{Fin}(n) \simeq \mathsf{Fin}(n)$$

Example: Cyclic lists

Cyclic lists come from a $\mathbb{Z}$-action on finite sets:

$$\mathsf{Cyc} = (n : \mathbb{N} \triangleright \mathsf{Fin}(n) \triangleleft^{\sigma_n} \mathbb{Z}) \qquad \sigma_n : \mathbb{Z} \to \mathfrak{S}(\mathsf{Fin}(n))$$

where σ_n is generated from the successor automorphism,

$$\begin{aligned} \mathsf{suc}_n &: \mathsf{Fin}(n) \simeq \mathsf{Fin}(n) \\ \mathsf{suc}_n(x) &:= x + 1 \mod n \end{aligned} \qquad \sigma_n(k) := \underbrace{\mathsf{suc}_n \circ \cdots \circ \mathsf{suc}_n}_{k \text{ times}}$$

In its interpretation,

$$[\![\mathsf{Cyc}]\!]X = \sum_{n:\mathbb{N}} X^n/\!\sim,$$

the relation $(\sim)$ is generated by

$$(x_1, \ldots, x_n) \sim (x_n, x_1, \ldots, x_{n-1})$$

Morphisms of action containers

A morphism of action containers preserves shapes, positions, *and* symmetries:

Definition

Let $F = (S \triangleright P \triangleleft^{\sigma} G)$, $G = (T \triangleright Q \triangleleft^{\tau} H)$. A morphism $(u \triangleright f \triangleleft \varphi) : F \to G$ consists of

- a function on shapes $u : S \to T$
- a family of functions $f : \prod_{s:S} Q_{us} \to P_s$
- a family of *group homomorphisms* $\varphi : \prod_{s:S} G_s \dot{\to} H_{us}$

such that the following diagram commutes for all $s : S$, $g : G_s$:

$$\begin{array}{ccc} Q_{us} & \xrightarrow{f_s} & P_s \\ {\scriptstyle \tau_{us}(\varphi_s(g))}\downarrow & & \downarrow{\scriptstyle \sigma_s(g)} \\ Q_{us} & \xrightarrow[f_s]{} & P_s \end{array}$$

Comparision to quotient containers I

Our definitions are inspired by *quotient containers*,[1] but there are differences:

Differences on objects

In a quotient container $(S \triangleright P/G)$, the symmetries are restricted to *subgroups of permutation groups*:

$$\iota_s : G_s \leq \mathfrak{S}(P_s)$$

Action containers can describe symmetry groups *larger* than $\mathfrak{S}(P_s)$ (e.g. the $\mathbb{Z}$-action in Cyc).

Comparision to quotient containers II

Differences on morphisms

- Morphisms of quotient containers do not respect symmetries: for all $g : G_s$ there merely exists *some* $h : H_{us}$ such that

$$\begin{array}{ccc} Q_{us} & \xrightarrow{f_s} & P_s \\ {\scriptstyle \iota'_{us}(h)}\downarrow & & \downarrow{\scriptstyle \iota_s(g)} \\ Q_{us} & \xrightarrow[f_s]{} & P_s \end{array}$$

  as opposed to the output of a group homomorphism $h \doteq \varphi(g)$.
- Morphisms of quotient containers are *quotiented* by some relation, such that $[\![-]\!]_{/} : \mathbf{QuotCont} \to \mathrm{Endo}\,(\mathbf{Set})$ is fully faithful.

[1] Abbott, Altenkirch, Ghani, and McBride, "Constructing Polymorphic Programs with Quotient Types".

Construction from universal property

The category of action containers is "just" families of group actions.

- For each shape $s : S$, there is some group (G_s) acting (σ_s) on some set (P_s).
- In a morphism, for each shape s there is some homomorphism (φ_s) and a function (f_s) that respect each other.

This suggests a modular construction.

A category of group actions

We define a category **Action** of *group actions* and *equivariant maps*:

- Objects are triples (G, P, σ), where σ is a G-action on P
- A morphism $(\varphi, f) : (G, P, \sigma) \to (H, Q, \tau)$ consists of
  - a group homomorphism $\varphi : G \dashrightarrow H$,
  - a function $f : P \leftarrow Q$,

  such that f is *equivariant*, i.e. for all $g : G$

$$\begin{CD} Q @>f>> P \\ @V{\tau(\varphi(g))}VV @VV{\sigma(g)}V \\ Q @>>f> P \end{CD}$$

- Morphisms compose by pasting equivariance squares.

This is like the category of G-sets, except G can vary.

Universal property

Definition (Fam-construction)

The *free coproduct completion* $\mathrm{Fam}(C)$ of a category C:

- **objects** pairs (S, x) of a set S and a family $x : S \to \mathrm{ob}(C)$
- **morphisms** for $(u, f) : (S, x) \to (T, y)$, a function $u : S \to T$ and a family of morphisms $f : \prod_{s:S} C(x_s, y_{us})$

Theorem

The category of action containers is equivalent to the free coproduct completion of the category of group actions:

$$\mathbf{ActionCont} \simeq \mathrm{Fam}(\mathbf{Action})$$

Closure properties

Presenting **ActionCont** as Fam(**Action**) gives us closure properties:

Proposition

Action containers are closed under (arbitrary) coproducts and products.

Proof.

- *coproducts: by construction*
- *products:* **Action** *is closed under products* □

Closure under exponentiation

Definition

The constant container at a set J is $\mathbf{k}J := (J \triangleright 0 \triangleleft^{\mathsf{id}} 1)$

Why constant? Answer: $[\![\mathbf{k}J]\!](X) \simeq J$.

Proposition

Action containers are closed under exponentiation by constants: For any container F and set J, there is a container F^J and a universal morphisms $\mathsf{eval} : \mathbf{k}J \times F^J \to F$.

A model of strictly positive types

Action containers model non-inductive single-variable strictly positive types.[2]

- strictly positive types are closed under products $F \times G$, coproducts $F + G$ and constant exponentiation F^J.
- single-variable: we do not consider *indexed* container[3]
- non-inductive: have not yet looked at smallest $\mu X.\, F(X, -)$ and largest $\nu X.\, F(X, -)$ fixedpoints yet.

[2] Abbott, Altenkirch, and Ghani, "Containers: Constructing strictly positive types".
[3] Conceptually not too hard, but requires more bookkeeping

Properties of the interpretation

Caveat

The interpretation functor is no longer fully faithful.

Reason

Quotients in **Set** forget why morphisms have been identified.
The evidence is an element of a symmetry group, G_s.

Fix

Make the evidence part of the data, and interpret action containers in *2-endofunctors of groupoids*.

Interpretation in groupoids

To interpret action containers in groupoids:

1. Enhance **ActionCont** into a 2-category, in small steps
2. Embed them into the 2-category of *symmetric containers*[4]
3. Compose with interpretation of the latter:

$$\mathbf{ActionCont} \longrightarrow \mathbf{SymmCont} \xhookrightarrow{[\![-]\!]} \mathsf{Endo}(\mathsf{hGpd})$$

[4] Gylterud, "Symmetric Containers".

A 2-category of action containers

Present the 2-category **ActionCont** again as families of group actions:

- **Group** forms a 2-category[5] with morphisms "up to conjugation":

$$\mathbf{Group}_2(\varphi, \psi) := \sum_{r:H} \varphi = r \psi r^{-1} \quad \forall \varphi, \psi : \mathbf{Group}_1(G, H)$$

  (r is called a *conjugator*)
- Similarly for **Action**: 2-cells are conjugators that relate equivariant maps.
- 2-cells of Fam C are families of 2-cells of C.

We define

$$\mathbf{ActionCont} := \mathrm{Fam}(\mathbf{Action})$$

[5] Hofstra and Karvonen, "Inner automorphisms as 2-cells".

Symmetric containers in HoTT

Definition

A *symmetric container* $(S \lhd P)$ consists of

shapes an *h-groupoid*[6] S

positions a *function* $P : S \to \mathrm{hSet}$

In HoTT, symmetries are internalized as paths:

- paths $s = t$ in S encode symmetries on shapes
- symmetries of positions are induced functorially:

$$\mathrm{cong}(P) : s = t \to P(s) = P(t)$$

- interpretation in h-groupoids:

$$[\![S \lhd P]\!] X := \sum_{s:S} P(s) \to X$$

[6] at most one proof $p = q$ for all $p, q : s = t$ in S

The Delooping-construction

A group G defines a 1-object h-groupoid $\mathbb{B}G$, implemented as a *higher inductive type*:

$$\frac{}{\bullet : \mathbb{B}G} \qquad \frac{g : G}{\mathsf{loop}\, g : \bullet = \bullet} \qquad \frac{g, h : G}{\mathsf{loop\text{-}comp}(g, h) : \mathsf{loop}\, g \cdot \mathsf{loop}\, h = \mathsf{loop}\, gh}$$

By recursion on $\mathbb{B}G$, each action $\sigma : G \dashrightarrow \mathfrak{S}(X)$ defines a family

$$\bar{\mathbb{B}}\sigma : \mathbb{B}G \to \mathsf{hSet}$$

Theorem

The above extend to 2-functors

$$\mathbb{B} : \mathbf{Group} \to \mathsf{hGpd} \qquad \bar{\mathbb{B}} : \mathbf{Action} \to \mathbf{SymmCont}$$

$$\bar{\mathbb{B}}(G, \sigma) := (\mathbb{B}G \lhd \bar{\mathbb{B}}\sigma)$$

Both are locally weak equivalences.

Symmetric containers from action containers I

We can embed single actions as symmetric containers:

$$\mathbf{Action} \xhookrightarrow{\bar{\mathbb{B}}} \mathbf{SymmCont}$$

We can lift this to families of such (i.e. action containers):

Theorem

The lifting $\mathsf{Fam}(\bar{\mathbb{B}}) : \mathbf{ActionCont} \to \mathsf{Fam}(\mathbf{SymmCont})$ *is*

1. *locally fully faithful*
2. *locally a weak equivalence, assuming the axiom of choice*

Symmetric containers from action containers II

What's missing?

$$\mathbf{ActionCont} \xhookrightarrow{\bar{\mathbb{B}}} \mathrm{Fam}(\mathbf{SymmCont}) \xrightarrow{???} \mathbf{SymmCont}$$

Proposition

There is a 2-functor, summing families of symmetric containers:

$$\Sigma : \mathrm{Fam}(\mathbf{SymmCont}) \to \mathbf{SymmCont}$$

The local functors

$$\Sigma_1 : \mathrm{Fam}_1(X, Y) \to \mathbf{SymmCont}_1(\Sigma_0 X, \Sigma_0 Y)$$

are fully faithful if all shape groupoids of X are connected.

Interpretation in groupoids

We can *locally* embed action containers in 2-endofunctors:

Theorem

The factorization

$$\mathbf{ActionCont} \xhookrightarrow{\mathsf{Fam}(\bar{\mathbb{B}})} \mathsf{Fam}(\mathbf{SymmCont}) \xrightarrow{\Sigma} \mathbf{SymmCont} \xhookrightarrow{[\![-]\!]} \mathsf{Endo}(\mathsf{hGpd})$$

is locally fully faithful.

Proof.

In the image of $\mathsf{Fam}(\bar{\mathbb{B}})$, *shape groupoids are connected.* □

This fully classifies 1- and 2-cells of action containers.

Conclusion

We have. . .

- constructed action containers via a universal property
- showed closure under desirable operations
- connected them to symmetric containers
- embedded them as 2-endofunctors
- developed all of this in *Cubical Agda*:
    - includes formalization of necessary 2-category theory

For a draft and the formalization:

`https://phijor.me/publications/2025-data-types-with-symmetries-via-action-containers.html`

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